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Path integral treatment of the gravitational anyon in a uniform magnetic field

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Abstract. The Green function for a relativistic particle interacting with a gravitational point source and a flux confined at the origin of the (ρ, ϕ)-space is evaluated using Feynman's summation-over-histories. The bound state energy spectrum is calculated when a uniform magnetic field is applied perpendicular to the (ρ, ϕ)-space.

1. Introduction

The discovery of anyons [1] as quantum states that are neither fermions nor bosons has evolved into a very active research area in recent years. The attraction to this field partly lies in the role of anyons in investigating high Tc superconductivity [2] and the fractional quantum Hall effect [3], as well as its link to a network of ideas that include the Aharonov–Bohm effect [4], cosmic strings [5], and $(2+1)$-dimensional gravity [6]. In fact, the presence of a gravitational anyon, in analogy to the electromagnetic case, has also been recently shown. The gravitational anyon can be exhibited with [7], or without [8] gravitational Chern–Simons term and it is the latter type that we consider in this paper. Specifically, we examine the system with a gravitational anyon and subject it to a uniform magnetic field. This generalized scenario is investigated here using the Feynman path integral [9].

We begin in section 2 by presenting a path integral approach to electromagnetic and gravitational anyons using the system discussed in [8], i.e. without a uniform magnetic field. The Green function for a relativistic particle interacting with a point source of spin σ and magnetic flux Φ is then evaluated as a summation-over-histories. The fractional angular momentum ℓ appears in a natural way as a δ-function constraint in the path integral. The Green function is obtained in closed form in section 3, in which the electromagnetic and gravitational anyons are contained as special cases.

In section 4, we generalize the system by introducing a uniform magnetic field perpendicular to the two spatial dimensions (ρ, ϕ). The Green function is evaluated for this case where the δ-function constraint in the path integral again yields the fractional angular momentum ℓ characterizing the electromagnetic and gravitational anyons. The presence of a uniform magnetic field gives rise to a harmonic oscillator type radial path integral and produces a bound state energy spectrum for the charge-flux (electromagnetic anyon) and the energy-spin (gravitational anyon) composites. The energy spectrum is obtained from the poles of the evaluated Green function.

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2. Electromagnetic and gravitational anyons: a unified approach

The gravitational anyon without the Chern–Simons term was demonstrated by Cho *et al.* [8] by considering the (2+1)-dimensional Einstein gravity and electrodynamics in a unified manner. This was done using a (3+1)-dimensional Kaluza–Klein metric given by

$$ds^2 = g_{ij}\,dx^i\,dx^j + (d\theta + A_i\,dx^i)^2 \tag{2.1}$$

where $g_{ij}(i, j = 0, 1, 2)$ is the (2+1)-dimensional spacetime metric, A_i is the electromagnetic potential, and θ is the 'internal' coordinate associated with the U(1) gauge group. Taking the two 'external' spatial coordinates as (ρ, ϕ), one introduces a magnetic flux Φ at the origin by choosing the potential, $A_i = (\Phi/2\pi)\,\partial_i\phi$. When g_{ij} in (2.1) describes a flat spacetime, this choice of A_i gives rise to the electromagnetic anyon (charge-flux composite) as discussed in [8]. The existence of the gravitational anyon, on the other hand, comes from the choice of g_{ij}. Specifically, the metric (2.1), with the flux Φ, can be written as [8]

$$ds^2 = -[dt + (\sigma/2\pi)\,d\phi]^2 + d\rho^2 + (1-\mu)^2\rho^2\,d\phi^2 + [d\theta + (\Phi/2\pi)\,d\phi]^2 \tag{2.2}$$

where σ is the spin and μ is the mass of the gravitational point source introduced at the origin of the (ρ, ϕ)-space [6].

One can apply the transformations,

$$t \to \tilde{t} = t + (\sigma/2\pi)\phi \tag{2.3a}$$

$$\phi \to \tilde{\phi} = (1-\mu)\phi \tag{2.3b}$$

$$\theta \to \tilde{\theta} = \theta + (\Phi/2\pi)\phi \tag{2.3c}$$

which recasts (2.2) into a flat form (except the singularity at the origin, $\rho = 0$) [6, 8]:

$$d\tilde{s}^2 = -d\tilde{t}^2 + d\rho^2 + \rho^2\,d\tilde{\phi}^2 + d\tilde{\theta}^2. \tag{2.4}$$

To provide a path integral treatment of the electromagnetic and gravitational anyons let us now consider a particle of mass M which interacts with the gravitational point source and flux Φ and obeys the Klein–Gordon equation

$$(\Box - M^2)G(x'', x') = -\delta(x'' - x') \tag{2.5}$$

where, $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$, and $g_{\mu\nu}$ is the metric of the spacetime described by (2.2), or (2.4) if one applies the transformations (2.3a)–(2.3c). The Green function $G(x'', x')$ can be expressed as the path integral [9, 10]

$$G(x'', x') = (\mathrm{i}/2\hbar)\int_0^\infty \exp[-\mathrm{i}M^2\Lambda/2\hbar]K(x'', x'; \Lambda)\,d\Lambda \tag{2.6}$$

where $K(x'', x'; \Lambda)$ is the propagator for the particle that goes from x' to x'' and is parametrized by a timelike variable λ $(0 < \lambda < \Lambda)$. This propagator can be written as an integral over all possible paths

$$K(x'', x'; \Lambda) = \int \exp[\mathrm{i}S/\hbar]\mathscr{D}[x] \tag{2.7}$$

where the action for the particle is $S = \int_0^\Lambda [(1/2)g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]\,d\lambda$, and $\dot{x}^\mu = dx^\mu/d\lambda$. We note that the propagator (2.7) satisfies a Schrödinger-like equation [9, 10]

$$\mathrm{i}\frac{\partial K}{\partial \Lambda} = -\Box K \tag{2.8}$$

where Λ serves as the time parameter and K satisfies the condition

$$\lim_{\Lambda \to 0} K(x'', x'; \Lambda) = \delta(x'' - x'). \tag{2.9}$$

The explicit evaluation of equations (2.6) and (2.7) for the electromagnetic and gravitational anyons is given in the following section.

3. Topologically constrained path integral

We shall now evaluate the propagator $K(x'', x'; \Lambda)$, equation (2.7), and use the result to calculate the Green function, $G(x'', x')$, (2.6). For convenience, the transformed metric (2.4) is taken as the starting point and the action S is then simply,

$$S = \int_0^\Lambda (1/2)[-\dot{\tilde{t}}^2 + \dot{\rho}^2 + \rho^2 \dot{\tilde{\phi}}^2 + \dot{\tilde{\theta}}^2]\,\mathrm{d}\lambda \tag{3.1}$$

where $\dot{x}^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$. Since the space being considered has a singularity at the origin of the $(\rho, \tilde{\phi})$-space, the paths, in going from $(\rho', \tilde{\phi}')$ to $(\rho'', \tilde{\phi}'')$, can be classified into homotopically inequivalent classes depending on the number of times n they wind, clockwise or anticlockwise, around the singularity [11, 12]. One can, therefore, decompose the propagator (2.7) in terms of the winding number n and write ($\tilde{x} = \tilde{t}, \rho, \tilde{\phi}, \tilde{\theta}$)

$$K(\tilde{x}'', \tilde{x}'; \Lambda) = \sum_{n=-\infty}^{\infty} K_n(\tilde{x}'', \tilde{x}'; \Lambda). \tag{3.2}$$

Let us first evaluate the case $n = 0$ (no winding). Following Feynman's prescription, we slice the timelike parameter Λ into N-subintervals, $\varepsilon = \lambda_j - \lambda_{j-1}$, and write K_0, as

$$\begin{aligned} K_0(\tilde{x}'', \tilde{x}'; \Lambda) &= \lim_{N\to\infty} \int \prod_{j=1}^{N} \exp\left[(\mathrm{i}/\hbar) \int_{\lambda_{j-1}}^{\lambda_j} \tfrac{1}{2}(-\dot{\tilde{t}}^2 + \dot{\rho}^2 + \rho^2\dot{\tilde{\phi}}^2 + \dot{\tilde{\theta}}^2)\,\mathrm{d}\lambda\right] \\ &\quad \times \prod_{j=1}^{N} (1/2\pi\mathrm{i}\hbar\varepsilon)^2 \prod_{j=1}^{N-1} (\mathrm{i}\rho_j\,\mathrm{d}\tilde{t}_j\,\mathrm{d}\rho_j\,\mathrm{d}\tilde{\phi}_j\,\mathrm{d}\tilde{\theta}_j) \\ &= \lim_{N\to\infty} \int \prod_{j=1}^{N} \exp\left[(\mathrm{i}/\hbar) \int_{\lambda_{j-1}}^{\lambda_j} \tfrac{1}{2}(\dot{\rho}^2 + \rho^2\dot{\tilde{\phi}}^2)\,\mathrm{d}\lambda\right] \\ &\quad \times K(\tilde{t}'', \tilde{t}')K(\tilde{\theta}'', \tilde{\theta}') \prod_{j=1}^{N} (1/2\pi\mathrm{i}\hbar\varepsilon) \prod_{j=1}^{N-1} (\rho_j\,\mathrm{d}\rho_j\,\mathrm{d}\tilde{\phi}_j). \end{aligned} \tag{3.3}$$

The propagator $K(\tilde{t}'', \tilde{t}')$, in (3.3), for the motion along the $\tilde{t}$-coordinate is similar in form to a free propagator, and hence path integrable, i.e.

$$\begin{aligned} K(\tilde{t}'', \tilde{t}') &= \lim_{N\to\infty} \int \prod_{j=1}^{N} \exp\left[(-\mathrm{i}/\hbar) \int_{\lambda_{j-1}}^{\lambda_j} (\tfrac{1}{2}\dot{\tilde{t}}^2)\,\mathrm{d}\lambda\right] \prod_{j=1}^{N} (1/2\pi\mathrm{i}\hbar\varepsilon)^{1/2} \prod_{j=1}^{N-1} (\mathrm{i}\,\mathrm{d}\tilde{t}_j) \\ &= (\mathrm{i}/2\pi) \int_{-\infty}^{+\infty} \exp[-\mathrm{i}E(\tilde{t}'' - \tilde{t}') + (\mathrm{i}\hbar E^2\Lambda/2)]\,\mathrm{d}E. \end{aligned} \tag{3.4}$$

On the other hand, the propagator $K(\tilde{\theta}'', \tilde{\theta}')$ in (3.3) for the 'internal' variable is similar to that of a particle on a circle [10]. The evaluation involves its decomposition in terms of winding numbers m and yields

$$K(\tilde{\theta}'', \tilde{\theta}') = \lim_{N\to\infty} \int \prod_{j=1}^{N} \exp\left[(\mathrm{i}/\hbar)\int_{\lambda_{j-1}}^{\lambda_j} (\tfrac{1}{2}\dot{\tilde{\theta}}^2)\,\mathrm{d}\lambda\right] \prod_{j=1}^{N} (1/2\pi\mathrm{i}\hbar\varepsilon)^{1/2} \prod_{j=1}^{N-1} (\mathrm{d}\tilde{\theta}_j)$$

$$= (2\pi\mathrm{i}\hbar\Lambda)^{-1/2} \sum_{m=-\infty}^{+\infty} \exp[\mathrm{i}(\tilde{\theta}'' - \tilde{\theta}' - 2\pi m)^2/2\hbar\Lambda]$$

which can be re-expressed as

$$K(\tilde{\theta}'', \tilde{\theta}') = (2\pi)^{-1} \sum_{e=-\infty}^{\infty} \exp[\mathrm{i}e(\tilde{\theta}'' - \tilde{\theta}') - (\mathrm{i}\hbar e^2\Lambda/2)]. \tag{3.5}$$

With (3.4) and (3.5), the propagator (3.3) can be written as

$$K_0(\tilde{x}'', \tilde{x}'; \Lambda) = [\mathrm{i}/(2\pi)^2] \int_{-\infty}^{+\infty} \mathrm{d}E \sum_{e=-\infty}^{\infty} \exp[-\mathrm{i}E(\tilde{t}'' - \tilde{t}') + \mathrm{i}e(\tilde{\theta}'' - \tilde{\theta}')]$$
$$\times \exp[\mathrm{i}(E^2 - e^2)\hbar\Lambda/2] K_0(\rho'', \tilde{\phi}''; \rho', \tilde{\phi}') \tag{3.6}$$

where

$$K_0(\rho'', \tilde{\phi}''; \rho', \tilde{\phi}')$$
$$= \lim_{N\to\infty} \int \prod_{j=1}^{N} \exp\left[(\mathrm{i}/\hbar)\int_{\lambda_{j-1}}^{\lambda_j} \tfrac{1}{2}(\dot{\rho}^2 + \rho^2\dot{\tilde{\phi}}^2)\,\mathrm{d}\lambda\right]$$
$$\times \prod_{j=1}^{N} (1/2\pi\mathrm{i}\hbar\varepsilon) \prod_{j=1}^{N-1} (\rho_j\,\mathrm{d}\rho_j\,\mathrm{d}\tilde{\phi}_j). \tag{3.7}$$

This type of path integral, (3.7), where a point singularity exists at the origin of the $(\rho, \tilde{\phi})$-space, has previously been examined [12]. Its path integration gives the result

$$K_0(\rho'', \tilde{\phi}''; \rho', \tilde{\phi}')$$
$$= (1/2\pi\mathrm{i}\hbar\Lambda) \exp[\mathrm{i}(\rho'^2 + \rho''^2)/2\hbar\Lambda]$$
$$\times \int_{-\infty}^{+\infty} \mathrm{d}\ell \exp[\mathrm{i}\ell(\tilde{\phi}'' - \tilde{\phi}')] I_{|\ell|}(\rho'\rho''/\mathrm{i}\hbar\Lambda) \tag{3.8}$$

where, $I(\rho'\rho''/\mathrm{i}\hbar\Lambda)$, is the modified Bessel function. Using equation (3.8) and the relations [13]

$$(1/2a)\cos\{[(b^2 + c^2)/4a] - (\nu\pi/2)\} J_\nu(bc/2a) = \int_0^\infty x \sin(ax^2) J_\nu(bx) J_\nu(cx)\,\mathrm{d}x \tag{3.9a}$$

for, $a > 0$, $b > 0$, $c > 0$, Re $\nu > -2$, where J_ν is the Bessel function, and

$$(1/2a)\sin\{[(b^2 + c^2)/4a] - (\nu\pi/2)\} J_\nu(bc/2a) = \int_0^\infty x \cos(ax^2) J_\nu(bx) J_\nu(cx)\,\mathrm{d}x \tag{3.9b}$$

for, $a > 0$, $b > 0$, $c > 0$, Re $\nu > -1$, the propagator (3.6) becomes

$$K_0(\tilde{x}'', \tilde{x}', \Lambda) = [\mathrm{i}/(2\pi)^3] \int_{-\infty}^{+\infty} \mathrm{d}E\,\mathrm{d}\ell \int_0^{+\infty} k\,\mathrm{d}k \sum_{\theta=-\infty}^{\infty} \exp[\mathrm{i}(E^2 - e^2 - k^2)\hbar\Lambda/2]$$
$$\times \exp[-\mathrm{i}E(\tilde{t}'' - \tilde{t}') + \mathrm{i}e(\tilde{\theta}'' - \tilde{\theta}') + \mathrm{i}\ell(\tilde{\phi}'' - \tilde{\phi}')] J_{|\ell|}(\rho' k) J_{|\ell|}(\rho'' k). \tag{3.10}$$

Having completed the path integration of K_0, we can express the propagator (3.10) in terms of the original coordinates, i.e. $(\tilde{t}, \rho, \tilde{\phi}, \tilde{\theta}) \to (t, \rho, \phi, \theta)$, using (2.3). This gives the propagator for $n = 0$ (no winding) in the original coordinates as

$$K_0(x'', x'; \Lambda) = [\mathrm{i}/(2\pi)^3] \int_{-\infty}^{+\infty} \mathrm{d}E\, \mathrm{d}\ell \int_0^{+\infty} k\, \mathrm{d}k \sum_{e=-\infty}^{\infty} \exp[\mathrm{i}(E^2 - e^2 - k^2)\hbar\Lambda/2]$$
$$\times \exp[-\mathrm{i}E(t'' - t') + \mathrm{i}e(\theta'' - \theta')]$$
$$\times \exp\{\mathrm{i}[-(E\sigma/2\pi) + (e\Phi/2\pi) + \ell(1-\mu)](\phi'' - \phi')\}$$
$$\times J_{|\ell|}(\rho' k) J_{|\ell|}(\rho'' k). \tag{3.11}$$

The expression for the propagator K_n for a given winding number n, can be obtained from (3.11) by simply replacing $(\phi'' - \phi')$ with $(\phi'' - \phi' + 2\pi n)$. Summing over all propagators with different winding numbers, the total propagator can be written as

$$K(x'', x'; \Lambda) = [\mathrm{i}/(2\pi)^3] \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}E\, \mathrm{d}\ell \int_0^{+\infty} k\, \mathrm{d}k \sum_{e=-\infty}^{\infty} \exp[\mathrm{i}(E^2 - e^2 - k^2)\hbar\Lambda/2]$$
$$\times \exp[-\mathrm{i}E(t'' - t') + \mathrm{i}e(\theta'' - \theta')]$$
$$\times \exp\{\mathrm{i}[-(E\sigma/2\pi) + (e\Phi/2\pi) + \ell(1-\mu)](\phi'' - \phi' + 2\pi n)\}$$
$$\times J_{|\ell|}(\rho' k) J_{|\ell|}(\rho'' k). \tag{3.12}$$

Using Poisson's summation formula

$$\sum_{n=-\infty}^{+\infty} \exp[\mathrm{i}n\Theta] = 2\pi \sum_{m=-\infty}^{+\infty} \delta(\Theta + 2\pi m) \tag{3.13}$$

equation (3.12) becomes

$$K(x'', x'; \Lambda) = [\mathrm{i}/(2\pi)^2] \int_{-\infty}^{+\infty} \mathrm{d}E\, \mathrm{d}\ell \int_0^{+\infty} k\, \mathrm{d}k \sum_{e=-\infty}^{\infty} \exp[\mathrm{i}(E^2 - e^2 - k^2)\hbar\Lambda/2]$$
$$\times \sum_{m=-\infty}^{+\infty} \delta[-E\sigma + e\Phi + 2\pi\ell(1-\mu) + 2\pi m]$$
$$\times \exp[-\mathrm{i}E(t'' - t') + \mathrm{i}e(\theta'' - \theta')]$$
$$\times \exp\{\mathrm{i}[-(E\sigma/2\pi) + (e\Phi/2\pi) + \ell(1-\mu)](\phi'' - \phi')\}$$
$$\times J_{|\ell|}(\rho' k) J_{|\ell|}(\rho'' k). \tag{3.14}$$

The integration over $\mathrm{d}\ell$ is facilitated by the δ-function in (3.14). This δ-function constraint, however, means that non-zero contributions to the total propagator comes from those with a fractional angular momentum ℓ given by

$$\ell = [(E\sigma/2\pi) - (e\Phi/2\pi) - m]/(1-\mu) \tag{3.15}$$

where, $m = 0, \pm 1, \pm 2, \ldots$. When $\sigma = \mu = 0$, one obtains the electromagnetic anyon (charge–flux composite) with the angular momentum

$$\ell = -(e\Phi/2\pi) + m \qquad m = 0, \pm 1, \pm 2, \ldots . \tag{3.16}$$

On the other hand, when $\Phi = \mu = 0$ in (3.14), the system exhibits a gravitational anyon (energy–spin composite) with a fractional angular momentum [8]

$$\ell = (E\sigma/2\pi) + m \qquad m = 0, \pm 1, \pm 2, \ldots . \tag{3.17}$$

From (3.14) the resulting $K(x'', x'; \Lambda)$ upon integration over $\mathrm{d}\ell$ can be written as

$$K(x'', x'; \Lambda) = \mathrm{i} \int_{-\infty}^{+\infty} \mathrm{d}E \int_0^{+\infty} k \, \mathrm{d}k \sum_{e,m=-\infty}^{\infty} \exp[\mathrm{i}(E^2 - e^2 - k^2)\hbar\Lambda/2] \times \Psi_{Ekme}(x'')\Psi^*_{Ekme}(x') \tag{3.18}$$

where the stationary wavefunction for the Λ-evolution is given by

$$\Psi_{Ekme}(x) = (1/2\pi)\exp[\mathrm{i}(-Et + e\theta + m\phi)] J_{|\frac{m\pm\alpha}{1-\mu}|}(\rho k) \tag{3.19}$$

and, $\alpha = (E\sigma/2\pi) - (e\Phi/2\pi)$. With (3.18) we can now proceed to evaluate the Green function (2.6) which involves integration over Λ. This gives

$$G(x'', x') = -\mathrm{i} \int_{-\infty}^{+\infty} \mathrm{d}E \int_0^{+\infty} k \, \mathrm{d}k \sum_{e,m=-\infty}^{\infty} \frac{\Psi_{Ekme}(x'')\Psi^*_{Ekme}(x')}{E^2 - e^2 - k^2 - (M/\hbar)^2 + \mathrm{i}\varepsilon} \tag{3.20}$$

where the limit $\varepsilon \to 0$ is implied, and the wave function is given by (3.19) where the coefficient becomes $(2\pi\hbar)^{-1}$. For the electromagnetic anyon ($\sigma = \mu = 0$), the wave function acquires the form found in [8].

4. Gravitational anyon in a uniform magnetic field

One can subject the electromagnetic and gravitational anyons discussed in the preceeding sections to a uniform magnetic field B by choosing the electromagnetic potential in (2.1) to be of the form

$$A_i = [(\Phi/2\pi) + (B\rho^2/2)]\,\partial_i\phi. \tag{4.1}$$

When the magnetic field B which is perpendicular to the (ρ, ϕ)-space is zero, then the system reduces to the one discussed in the previous sections. With (4.1) plus a gravitational point source, (2.1) becomes

$$\mathrm{d}s^2 = -[\mathrm{d}t + (\sigma/2\pi)\,\mathrm{d}\phi]^2 + \mathrm{d}\rho^2 + (1-\mu)^2\rho^2\,\mathrm{d}\phi^2 + \{\mathrm{d}\theta + [(\Phi/2\pi) + (B\rho^2/2)]\,\mathrm{d}\phi\}^2. \tag{4.2}$$

Application of the transformation

$$\mathrm{d}t \to \mathrm{d}\tilde{t} = \mathrm{d}t + (\sigma/2\pi)\,\mathrm{d}\phi \tag{4.3a}$$

$$\mathrm{d}\phi \to \mathrm{d}\tilde{\phi} = (1-\mu)\,\mathrm{d}\phi \tag{4.3b}$$

$$\mathrm{d}\theta \to \mathrm{d}\tilde{\theta} = \mathrm{d}\theta + [(\Phi/2\pi) + (B\rho^2/2)]\,\mathrm{d}\phi \tag{4.3c}$$

reduces (4.2) to a flat form given by (2.4) (except for the singularity at the origin). We can then consider a relativistic particle obeying the Klein–Gordon equation (2.5) where the Green function $G(x'', x')$ can be written as the path integral (2.6) with (2.7). Using the flat form of the metric, the propagator can be decomposed in terms of the winding numbers as in (3.2) in view of the singularity at the origin. The path integration for the $\tilde{t}$ and $\tilde{\theta}$ variables proceeds as in section 3, and we obtain for zero winding ($n = 0$)

$$K_0(\tilde{x}'', \tilde{x}'; \Lambda) = [\mathrm{i}/(2\pi)^2] \int_{-\infty}^{+\infty} \mathrm{d}E \sum_{e=-\infty}^{\infty} \exp[\mathrm{i}(E^2 - e^2)\hbar\Lambda/2] \times \exp[-\mathrm{i}E(\tilde{t}'' - \tilde{t}') + \mathrm{i}e(\tilde{\theta}'' - \tilde{\theta}')]$$

$$\times \lim_{N\to\infty} \int \prod_{j=1}^{N} \exp\left[(\mathrm{i}/\hbar) \int_{\lambda_{j-1}}^{\lambda_j} \tfrac{1}{2}(\dot\rho^2 + \rho^2 \dot{\tilde\phi}^2)\, \mathrm{d}\lambda \right]$$

$$\times \prod_{j=1}^{N} (1/2\pi \mathrm{i}\hbar\varepsilon) \prod_{j=1}^{N-1} (\rho_j\, \mathrm{d}\rho_j\, \mathrm{d}\tilde\phi_j). \tag{4.4}$$

At this stage, only the component for the $(\rho, \tilde\phi)$-space needs to be path integrated. We also note that $\tilde\theta$, unlike (2.3c), now depends on ρ and the path integration over $\mathrm{d}\rho$ has to account for this. From (4.3a)-(4.3c), the terms $(\tilde t'' - \tilde t')$, $(\tilde\theta'' - \tilde\theta')$, and $\tilde\phi$ in (4.4) can be expressed, respectively, as

$$\tilde t'' - \tilde t' = t'' - t' + (\sigma/2\pi)(\phi'' - \phi') \tag{4.5a}$$

$$\tilde\theta'' - \tilde\theta' = \theta'' - \theta' + \int [(\Phi/2\pi) + (B\rho^2/2)]\, \mathrm{d}\phi \tag{4.5b}$$

$$\tilde\phi = (1-\mu)\phi \tag{4.5c}$$

We observe that in transforming back to the original coordinates, (t, ρ, ϕ, θ), equation (4.5a) is already in terms of the endpoints $(t'', \phi''; t', \phi')$ and involves no further integration (the same situation occurs in section 3). Equation (4.5b), however, differs from section 3 due to a non-vanishing B-field. With (4.5), equation (4.4) acquires the form

$$K_0(x'', x'; \Lambda) = [\mathrm{i}/(2\pi)^2] \int_{-\infty}^{+\infty} \mathrm{d}E \sum_{e=-\infty}^{\infty} \exp[\mathrm{i}(E^2 - e^2)\hbar\Lambda/2]$$

$$\times \exp[-\mathrm{i}E(t'' - t') + \mathrm{i}e(\theta'' - \theta')]$$

$$\times \exp\{\mathrm{i}[-(E\sigma/2\pi) + (e\Phi/2\pi)](\phi'' - \phi')\} K_0(\rho'', \phi''; \rho', \phi') \tag{4.6}$$

where

$$K_0(\rho'', \phi''; \rho', \phi')$$

$$= \lim_{N\to\infty} \int \prod_{j=1}^{N} \exp[(\mathrm{i}/\hbar) S_j(\rho, \phi)] \prod_{j=1}^{N} (1/2\pi\mathrm{i}\hbar\varepsilon) \prod_{j=1}^{N-1} [\rho_j\, \mathrm{d}\rho_j (1-\mu)\, \mathrm{d}\phi_j] \tag{4.7}$$

with the short-time action (let $\beta = \hbar e B/4$, which is a constant)

$$S_j(\rho, \phi) = \int_{\lambda_{j-1}}^{\lambda_j} \tfrac{1}{2}(\dot\rho^2 + \rho^2(1-\mu)^2\dot\phi^2 + 2\beta\rho^2\dot\phi)\, \mathrm{d}\lambda. \tag{4.8}$$

The path integration of (4.7), with (4.8), can be simplified by introducing a new angular variable γ defined by

$$\gamma = (1-\mu)\phi + (\beta/1-\mu)\lambda \tag{4.9}$$

such that (note: $\dot x^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$)

$$\dot\gamma^2 - (\beta/1-\mu)^2 = (1-\mu)^2\dot\phi^2 + 2\beta\dot\phi. \tag{4.10}$$

With (4.10) the short-time action (4.8) becomes

$$S_j(\rho, \gamma) = \int_{\lambda_{j-1}}^{\lambda_j} \tfrac{1}{2}(\dot\rho^2 + \rho^2\dot\gamma^2 - \omega^2\rho^2)\, \mathrm{d}\lambda \tag{4.11}$$

where, $\omega^2=(\beta/1-\mu)^2$. We also note that with (4.9), we have, $\gamma_j=\gamma(\varepsilon_j)=(1-\mu)\phi_j+(\beta/1-\mu)\varepsilon_j$, where $\varepsilon_j=\lambda_j-\lambda_{j-1}=$ constant, since Λ is sliced into N equal sub-intervals. Hence, for the measure we have, $\mathrm{d}\gamma_j=(1-\mu)\,\mathrm{d}\phi_j$, or the relation

$$\prod_{j=1}^{N-1}[\rho_j\,\mathrm{d}\rho_j(1-\mu)\,\mathrm{d}\phi_j]=\prod_{j=1}^{N-1}[\rho_j\,\mathrm{d}\rho_j\,\mathrm{d}\gamma_j]. \tag{4.12}$$

With (4.11) and (4.12), the propagator (4.7) can be written as

$$K_0(\rho'',\gamma'';\rho',\gamma')=\lim_{N\to\infty}\int\prod_{j=1}^{N}\exp[(\mathrm{i}/\hbar)S_j(\rho,\gamma)]\prod_{j=1}^{N}(1/2\pi\mathrm{i}\hbar\varepsilon)\prod_{j=1}^{N-1}[\rho_j\,\mathrm{d}\rho_j\,\mathrm{d}\gamma_j], \tag{4.13}$$

with the short-time action [12]

$$S_j(\rho,\gamma)=(\Delta\rho_j)^2/2\varepsilon+\hat{\rho}_j^2(\Delta\gamma_j)^2/2\varepsilon+\hbar^2\varepsilon/8\hat{\rho}_j^2-\omega^2\hat{\rho}_j^2\varepsilon/2 \tag{4.14}$$

where, $\Delta\rho_j=\rho_j-\rho_{j-1}$, $\Delta\gamma_j=\gamma_j-\gamma_{j-1}$, and $\hat{\rho}_j^2=\rho_j\rho_{j-1}$. We can extract the angular part from (4.13) and (4.14) and path integrate in the universal covering space [11] where $-\infty<\gamma<+\infty$. The angular part $A(\gamma'',\gamma')$ gives [12]

$$\begin{aligned}A(\gamma'',\gamma')&=\lim_{N\to\infty}\int\prod_{j=1}^{N}\exp[\mathrm{i}\hat{\rho}_j^2(\Delta\gamma_j)^2/2\hbar\varepsilon]\prod_{j=1}^{N}(1/2\pi\mathrm{i}\hbar\varepsilon)^{1/2}\prod_{j=1}^{N-1}[\rho_j\,\mathrm{d}\gamma_j]\\&=[2\pi(\rho'\rho'')^{1/2}]^{-1}\int_{-\infty}^{\infty}\exp[\mathrm{i}\ell(\gamma''-\gamma')]\\&\quad\times\prod_{j=1}^{N}\exp[-\mathrm{i}\hbar\ell^2\varepsilon/2\hat{\rho}_j^2]\,\mathrm{d}\ell.\end{aligned} \tag{4.15}$$

With (4.15), equation (4.13) becomes,

$$K_0(\rho'',\gamma'';\rho',\gamma')=(2\pi)^{-1}\int_{-\infty}^{\infty}\mathrm{d}\ell\,\exp[\mathrm{i}\ell(\gamma''-\gamma')]K_\ell(\rho'',\rho') \tag{4.16}$$

where only the radial part $K_\ell(\rho'',\rho')$ needs to be path integrated

$$\begin{aligned}K_\ell(\rho'',\rho')&=(\rho''\rho')^{-1/2}\\&\quad\times\lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\{(\mathrm{i}/\hbar)[(\Delta\rho_j)^2/2\varepsilon-(\ell^2-1/4)\hbar^2\varepsilon/2\hat{\rho}_j^2-\omega^2\hat{\rho}_j^2\varepsilon/2]\}\\&\quad\times\prod_{j=1}^{N}(1/2\pi\mathrm{i}\hbar\varepsilon)^{1/2}\prod_{j=1}^{N-1}[\mathrm{d}\rho_j].\end{aligned} \tag{4.17}$$

Equation (4.17) is just the radial path integral for the harmonic oscillator whose evaluation yields

$$\begin{aligned}K_\ell(\rho'',\rho')&=(\omega/\mathrm{i}\hbar)\operatorname{cosec}(\omega\Lambda)\exp[(\mathrm{i}\omega/2\hbar)(\rho'^2+\rho''^2)\cot(\omega\Lambda)]\\&\quad\times I_{|\ell|}[(\omega/\mathrm{i}\hbar)\rho''\rho'\operatorname{cosec}(\omega\Lambda)].\end{aligned} \tag{4.18}$$

With (4.6), (4.16) and (4.18), the path integrated propagator for zero winding ($n=0$) is,

$$K_0(x'',x';\Lambda)=[\mathrm{i}/(2\pi)^3]\int_{-\infty}^{+\infty}\mathrm{d}E\,\mathrm{d}\ell\sum_{e=-\infty}^{\infty}\exp[-\mathrm{i}E(t''-t')+\mathrm{i}e(\theta''-\theta')]$$

$$\times \exp\{i[-(E\sigma/2\pi)+(e\Phi/2\pi)+\ell(1-\mu)](\phi''-\phi')\}$$
$$\times \exp\{i[E^2-e^2+(2\ell\beta)/(1-\mu)\hbar]\hbar\Lambda/2\}$$
$$\times (\omega/i\hbar)\,\mathrm{cosec}(\omega\Lambda)\exp[(i\omega/2\hbar)(\rho'^2+\rho''^2)\cot(\omega\Lambda)]$$
$$\times I_{|\ell|}[(\omega/i\hbar)\rho''\rho'\,\mathrm{cosec}(\omega\Lambda)]. \tag{4.19}$$

To obtain the form of the propagator K_n for a given winding number n, we replace the term $(\phi''-\phi')$ in (4.19) by $(\phi''-\phi'+2\pi n)$. Summing over all K_ns, as in (3.2), the propagator (2.7) can now be written as

$$K(x'',x';\Lambda)=[i/(2\pi)^2]\int_{-\infty}^{+\infty}\mathrm{d}E\,\mathrm{d}\ell\sum_{e=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\delta[-E\sigma+e\Phi+2\pi\ell(1-\mu)+2\pi m]$$
$$\times \exp[-iE(t''-t')+ie(\theta''-\theta')]$$
$$\times \exp\{i[-(E\sigma/2\pi)+(e\Phi/2\pi)+\ell(1-\mu)](\phi''-\phi')\}$$
$$\times \exp\{i[E^2-e^2+(2\ell\beta)/(1-\mu)\hbar]\hbar\Lambda/2\}$$
$$\times (\omega/i\hbar)\,\mathrm{cosec}(\omega\Lambda)\exp[(i\omega/2\hbar)(\rho'^2+\rho''^2)\cot(\omega\Lambda)]$$
$$\times I_{|\ell|}[(\omega/i\hbar)\rho''\rho'\,\mathrm{cosec}(\omega\Lambda)]. \tag{4.20}$$

In (4.20), we applied Poisson's summation formula to write

$$\sum_{n=-\infty}^{\infty}\exp\{i[-E\sigma+e\Phi+2\pi\ell(1-\mu)]n\}$$
$$=(2\pi)\sum_{m=-\infty}^{\infty}\delta[-E\sigma+e\Phi+2\pi\ell(1-\mu)+2\pi m]. \tag{4.21}$$

The δ-function in (4.20) acts as a constraint that yields a fractional angular momentum given by (3.15). From this the angular momenta for the electromagnetic anyon, (3.16), and gravitational anyon, (3.17), can again be obtained as special cases.

Integrating the variable ℓ, the resulting $K(x'',x';\Lambda)$ can be used to evaluate the Green function, (2.6), which now acquires the form

$$G(x'',x')=(2i)^{-1}/(2\pi i\hbar)^2\int_{-\infty}^{+\infty}\mathrm{d}E\sum_{e,m=-\infty}^{\infty}\exp[-iE(t''-t')$$
$$+ie(\theta''-\theta')+im(\phi''-\phi')]$$
$$\times\int_0^{+\infty}\exp\{i[E^2-e^2-(M/\hbar)^2+2\beta(m+\alpha)/(1-\mu)^2\hbar]\hbar\Lambda/2\}$$
$$\times\mathrm{cosec}(\omega\Lambda)\exp[(i\omega/2\hbar)(\rho'^2+\rho''^2)\cot(\omega\Lambda)]$$
$$\times I_{|\frac{m+\alpha}{1-\mu}|}[(\omega/i\hbar)\rho''\rho'\,\mathrm{cosec}(\omega\Lambda)]\omega\,\mathrm{d}\Lambda \tag{4.22}$$

where $\beta=(\hbar eB/4)$, $\alpha=(E\sigma/2\pi)-(e\Phi/2\pi)$. The last three factors represent the radial part of a symmetric harmonic oscillator propagator corresponding to the fractional angular momentum, $\ell=|(m+\alpha)/(1-\mu)|$. We can, therefore, write

$$[\omega/i\sin(\omega\Lambda)]\exp[(i\omega/2\hbar)(\rho'^2+\rho''^2)\cot(\omega\Lambda)]I_\ell[(\omega/i\hbar)\rho''\rho'\,\mathrm{cosec}(\omega\Lambda)]$$
$$=\sum_{n_r=0}^{\infty}R_{n_r\ell}(\rho'')R_{n_r\ell}(\rho')\exp[-i(2n_r+\ell+1)\omega\Lambda] \tag{4.23}$$

where $n_r = 0, 1, 2, \ldots$, is a radial quantum number and $R_{n_r\ell}(\rho)$ are normalized eigenfunctions in terms of the generalized Laguerre polynomials $L_n^\ell(x)$

$$R_{n_r\ell}(\rho) = [(2n_r!)\omega^{\ell+1}/\Gamma(n_r+\ell+1)]^{1/2}(\rho/\hbar)^{\ell/2}\exp(-\omega\rho^2/2\hbar)L_{n_r}^{\ell}(\omega\rho^2/\hbar) \tag{4.24}$$

Inserting (4.23) into (4.22) and integrating over Λ we arrive at the Green function

$$G(x'', x') = (1/\mathrm{i}\hbar)\int_{-\infty}^{+\infty} \mathrm{d}E \sum_{e,m=-\infty}^{\infty}\sum_{n_r=0}^{\infty} \frac{\Psi_{En_rme}(x'')\Psi^*_{En_rme}(x')}{\kappa^2-(2n_r+\ell+1)(2\omega/\hbar)+\mathrm{i}\varepsilon} \tag{4.25}$$

where the limit $\varepsilon \to 0$ is implied, $\kappa^2 = E^2 - e^2 - (M/\hbar)^2 + 2\omega(m+\alpha)/(1-\mu)\hbar$, and

$$\Psi_{En_rme}(x) = (1/2\pi\hbar)\exp[\mathrm{i}(-Et + e\theta + m\varphi)]R_{n_r\ell}(\rho). \tag{4.26}$$

The energy levels are obtained from the Green function as poles in the variables κ^2 in the points

$$\kappa^2 = (2n_r + \ell + 1)(2\omega/\hbar). \tag{4.27}$$

Equation (4.27) yields the energy spectrum,

$$E = \pm\{(n_r + 1/2)eB/(1-\mu) + e^2 + (M/\hbar)^2\}^{1/2}. \tag{4.28}$$

5. Conclusion

In this paper, we have presented the following:

(i) a unified path integral treatment of the gravitational and electromagnetic anyons where the fractional angular momentum manifests as a δ-function constraint; and

(ii) an investigation of the gravitational and electromagnetic anyons in the presence of a uniform magnetic field where a bound state energy spectrum is obtained.

We also note that the above evaluation provides another example of an exactly path integrable system in the realm of Kaluza–Klein theories [14].

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